Paper: Introductory Mathematical Methods for Economics

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Paper : DSC 2 : Introductory Mathematical Methods for Economics

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All questions are compulsory. PwD marked questions are alternatives to be attempted ony by PwD students.

Q. 1. Answer any two of the following: 2×4=8

(a) (i) Find all values if x satisfying $(|x| - 2)(x + 5) < 0$.

(ii) If A and B are two sets containing 4 and 7 elements respectively, find the maximum and minimum number of elements in $A \cup B$.

Ans. (i) To find the value of x satisfying the inequality $(|x| - 2)(x + 5) < 0$, we meed to analyze the signs of both factors on the left hand side of the inequality.

First, let's consider the factor $|x| - 2$.

- When $x \geq 0$, $|x| - 2$ is positive
- When $x < 0$, $|x| - 2$ is negative.

Next, let's consider the factor $x + 5$.

- When $x > -5$, $x + 5$ is positive.
- When $x < -5$, $x + 5$ is negative.

To satisfy the inequality $(|x| - 2)(x + 5) < 0$, we need one factor to be negative and the other factor to be positive.

Case 1: $|x| - 2 < 0$ and $x + 5 > 0$

The implies $|x| > 2$ and $x < -5$, which simplifies to $x \in (-2, 2)$.

Case 2: $|x| - 2 > 0$ and $x + 5 < 0$

This implies $|x| > 2$ and $x < -5$, when simplifies to $x \in (-\infty, -5)$

So, the solution for x that satisfies $(|x| - 2)(x + 5) < 0$ is $x \in (-\infty, -5) \cup (-2, 2)$.

(ii) The maximum number of elements in the union of sets A and B occurs when there is no overlap between the two sets. In this case, the maximum number of elements is the sum of the lements in A and B.

$$\text{Maximum} = 4 + 7 = 11$$

The minimum number of elements in the union of sets A and B occurs when they have maximum overlap. In this case, the minimum number of elements is the larger of the two set sizes.

$$\text{Minimum} = \text{Max}(4, 7) = 7.$$

So , the maximum number of elements in $A \cup B$ is 11 and the minimun number of elements is 7

(b) Fill in the blank with necessary, sufficient of necessary and sufficient:

(i) If A is a sufficient condition for B, then $\sim B$ is ______ condition for $\sim A$.

(ii) For a rectangle to be considered a square, having four sides of equal length is ______ condition.

(iii) $x > 0$ is ______ for $x(x + 4) > 0$.

(iv) For two sets X and Y, $X \cup Y = X$ is ______ condition for Y to be a subset of X.

Ans. (i) If A is a sufficient condition for B, then $\sim B$ is ***necessary*** condition for $\sim A$.

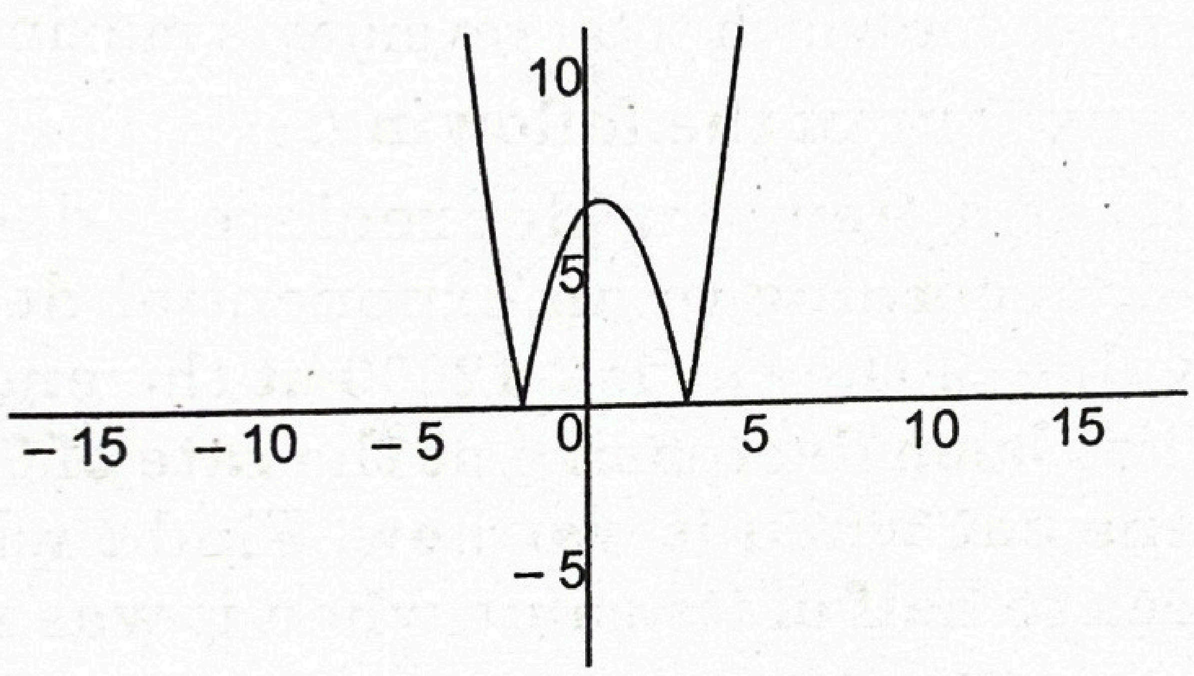
(ii) For a rectangle to be considered a square, having four sides of equal length is ***both necessary and suffcent*** condition.

(iii) $x > 0$ is ***sufficient*** for $x(x + 4) > 0$.

(iv) For two sets X and Y, $X \cup Y = X$ is ***sufficient*** condition for Y to be a subset of X.

(c) Graph $f(x) = |x^2 - x - 6|$.

Ans. The graph is as follows of $f(x) = |x^2 - x^{-6}|$



PwD (c) Suppose the consumers of a product demand 60 uinits of a product when the price is ₹ 5 per unit and 40 units when the price per unit has gone up by ₹ 4.

(i) Find the demand equation foe the product, assuming that it is linear.

(ii) Express total revenue as a function of price and find the price for which total revenue is maximum.

Ans. (i) To find the demand equation for the product, assuming that it is linear, we can use the two given points as data: (Price, Demand). Let's use the points as data (5, 60) and (9, 40) since the price has gone up by Rs. 4.

We'll use the point-slope form of a linear equation:

$$y - y_1 = m(x - x_1), \text{ where m is the slope.}$$

Let's use the point (5, 60):

$$m = \frac{y - y_1}{x - x_1} = \frac{40 - 60}{9 - 5} = -5$$

Now that we have the slope, we can use it to find the equation:

$$y - 60 = -5(x - 5)$$
$$y = -5x + 85$$

So, the demand equation for the product is $D(x) = -5x + 85$, where $D(x)$ is the demand for x units of the product and x is the price per unit.

(ii) Total revenue (TR) is calculated by multiplying the price per unit (x) by the quantity demanded $D(x)$:

$$TR = x \cdot D(x)$$

From the demand equation we found earlier, $D(x) = -5x + 85$, so

Hence, the total revenue TR is a quadratic function of the price per unitx, given by $TR = -5x^2 + 85x$.

To find the price for which total revenue is maximum, we need to find the vertex of the quadratic function $TR = -5x^2 + 85x$. The x-coordinate of the vertex can be found using the formula:

$$x_{vertex} = -b/2a$$

In this case,

$$a = -5 \text{ and } b = 85, \text{ so}$$

$$x_{vertex} = -85/2(-5) = 8.5$$

Therefore, the price for which total revenue is maximum is ₹ 8.5 per unit.

Q. 2. Answer any four of the following: **4×4=16**

(a) The value of a new car depreciates (decreases) after it is purchased, according to an exponential decay model. Suppose that the value of the car is ₹ 12000 at the end of 5 years and that its value has been decreasing at the rate of 9% per year. Find the value of the car when it was new. Find *t* when the value of the car reduces to half of its value when it was new.

Ans. Let y be the valie of the car after t years: $y = ab^t$, $r = -0.09$ and $b = 1 + r + 1 + (-0.09) = 0.91$.

The function is $y = a(0.91)^t$

In this case we know that when $t = 5$, then $y = 12000$; substituting these values gives

$$12000 = a(0.91)^5$$

We need to solve for the initial value a, the purchase price of the car when new. First evaluate $(0.91)^5$, then solve the resulting linear equation to find a.

$$12000 = a(0.624)$$

$a = 12000/0.624 =$ ₹ 19230.77; The car's value was ₹ 19, 230.77 when it was new.

Next, let's find the time t when the value of the car reduces to half of its value when is was new:

We need to solve for t in the equation:

$$\frac{V_0}{2} = V_0 \times e^{-0.09 \times t}$$

Simplifying:

$$e^{-.09 \times t} = \frac{1}{2}$$

Taking the natural logarithm of both sides:

$$-0.09 \times t = \ln\left(\frac{1}{2}\right)$$

Solving for t:

$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.09}$$

Using properties of logarithma and the valie of e:

$$t \approx \frac{-\ln(2)}{0.09} \approx 7.72$$

So, it take approximately 7.72 years for the value of the car to reduce to half of its value when it was new.

(b) A country exports three goods, wheat W, coal C and palm oil 0. At time $t = t_0$, the revenue in crores of rupees derived from each of these goods is $W(t_0) = 4$, $C(t_0) = 10$ and $0(t_0) = 7$. W is declining at 3% while 0 and C are growing at 15% and 8% respectively. Find the rate of growth of total export earnings at $t = t_0$.

Ans. The total export earnings at time t is the sum of the revenues from each of the three goods: wheet (W), Coal (C), and palm oil (O). Let's denote the total export earnings as $E(t)$, which is given by:

$$E(t) = W(t) + C(t) + O(t)$$

Given the initial revenues at $t = t_0$, we have

$$W(t_0) = 4$$
$$C(t_0) = 10$$
$$O(t_0) = 7$$

The growth rates are given as follows:

- Wheat (W) is declining at 39% per year, which means its rate of change is – 0.09.
- Coal (C) is growing at 89% per year, which means its rate of change is 0.08.
- Palm oil (O) is growing at 15% per year, which means its rate of change is 0.15.

To find the rate of growth of the total export earnings at $t = t_0$, we can calculate the derivative of $E(t)$ with respect to t and then evaluate it at $t = t_0$:

$$E'(t) = W'(t) + C'(t) + O'(t)$$

Plugging in the given growth rates:

$$E'(t) = -0.03 \cdot W(t) + 0.08 \cdot C(t) + 0.15 \cdot O(t)$$

At $t = t_0$, we have

$$E'(t_0) = -0.03 \cdot W(t_0) + 0.08 \cdot C(t_0) + 0.15 \cdot O(t_0)$$

Substitute the given initial values:

$$E'(t_0) = -0.03 \cdot 4 + 0.08 \cdot 10 + 0.15 \cdot 7$$

Calculate the values:

$$E'(t_0) = -0.12 + 0.8 + 1.05 = 1.73$$

So, the rate of growth of total export earnings at $t = t_0$ is approximately 1.73 crore rupees per year.

(c) Examine the inverse demand curve $p = \frac{20}{x+1}$. Show that the demand increases from 0 to indefinitely large amounts as price falls. Find total revenue and show that it increases to a limiting value.

Ans. The inverse demand curve is given by $P = \frac{20}{x+1}$, where P represents the price and x represents the quantity demanded. let's analyze the behaviour of this inverse demand curve:

1. As P (price) decreases, the denominator $x + 1$ increases, and thus the value of x (quantitu demanded) increaases. This means that as the price falls, the quantity demanded increases. This behaviour is consistent with the law of demand, which states that, all else being equal, as the price of a good decreases, the quantity demanded increases.
2. As P approaches O (falls toweards O), the value of x increases towards infinity. The implies that the quantity demanded can increase to indefinitely large amounts as the price approaches zero. However, in practice, there might be practical constraints that precent the price from reaching zero or quantities from becoming infinitely large.

Now, let's find the total revenues and show that it increases to a limiting value.

Total revenue (TR) is calculated by multiplying the price IP) by the quantity demanded (x):

$$TR = P \,.\, z = \frac{20\,x}{x+1}$$

To analyze the behaviour of total revenue as x increases, we can take the derivative of TR with respect to x and analyze its sign:

$$TR' = \frac{d\,(TR)}{dx} = \frac{20}{(x+1)^7}$$

The derivative TR' is always positive because the denominator $(x + 1)^2$ is positive regardless of the value of x. This indicates that the total revenue is increasing as x increases. In other words, as the quantity demanded increases, the total revenue also increases.

Howerver, as x becomes very large, the effect of the denominator $(x + 1)^2$ becomes relatively smaller, and the rate of increase in total revenue slows down. This implies that total revenue increases to a limiting value as x gbecomes vary large.

In summary, the inverse demand curve shows that as the price falls, the quantity demanded increases, and total revenue also increases, reaching a limiting value as quantities become vary large. This behaviour is in line with economic principles and the concepts of demand and total revenue.

(d) Consider an infinite series $\sum_{i=1}^{\infty} a_i$. Prove that $\lim_{n \to \infty} a_n = 0$ is necessary for convergence for the series, but not sufficient.

Ans. To prove that $\lim_{n \to \infty} a_n = 0$ is necessary for the convergence of an infinite series $\sum_{i=1}^{\infty} a_i$, will use the Contrapositive Convergence Tesat. This test states that if the terms of the series a_n do not approach zero as n goes to infinity (i.e., $\lim_{n \to \infty} a_n \neq 0$), then the series $\sum_{i=1}^{\infty} a_i$, must diverge.

To show that $\lim_{n \to \infty} a_n = 0$ is not sufficient for the convergence of the seriesm we'll provide a counterexample.

Proof that $\lim_{n \to \infty} a_n = 0$ is necessary for convergences:

Contrapsitive Convergence Test: if $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{n \to \infty} a_n = 0$.

The contrapositive statement is as follows:

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{i=1}^{\infty} a_i$ diverges.

This contrapositive statement is essentially the Contrapositive Convergence Test. which show that if the terms do not approach zero, the series must diverge.

Proof that $\lim_{n \to \infty} a_n = 0$ is not sufficient for convergence:

Consider the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$. This services is known to be the harmonic series, and if diverges. However, $\lim_{n \to \infty} \frac{1}{n} = 0$.

This example illustrates that even though the terms approach zero, the series still diverges.

In summary

- The $\lim_{n \to \infty} a_n = 0$ is necessary for the convergence of an infinite series $\sum_{i=1}^{\infty} a_i$.
- However, the limit $\lim_{n \to \infty} a_n = 0$ is not sufficient for the convergence of the series.

There exist series where the terms approach zero, but the series diverges.

The harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$ is a clasic examples of the case.

(e) If $f(x) = \frac{x^n}{e^x}$, show that $f(x)$ decreases for $x \geq n > 0$ and find the local maximum value of $f(x)$. Find $f(2x)$ and show that $\frac{2^n x^n}{e^x e^x} \leq \frac{2^n n^n}{e^n e^x}$.

Ans. To prove that $f(x) = \frac{x^n}{e^x}$ decrease for $x \geq n > 0$, find the localmaximum value of $f(x)$, and demonstrate the given inequality, we'll go through each step.

Proving that $f(x)$ decreases for $x \geq n > 0$:

To show that $f(x)$ decreases for $x \geq n > 0$, we need to prove that its derivatives is negative for the interval.

Given: $$f(x) = \frac{x^n}{e^x}$$

Find the derivative $f'(x)$:

$$f(x) = \frac{x^n}{e^x}$$

$$f'(x) = \frac{nx^{n-1} e^x - x^n e^x}{e^{2x}}$$

For $x \geq n > 0$, all terms in the numerator of $f'(x)$ are positive, and the denominator is always positive. Therefore, $f'(x) > 0$, which means $f(x)$ is increasing for $x \geq n > 0$.

Finding the total maximum cvalued of $f(x)$:

The local maximum occurs where the derivatives changes sign from positive to negative.

We'll find where $f'(x) = 0$

$$f'(x) = \frac{nx^{n-1} e^x - x^n e^x}{e^{2x}} = 0$$

Solving for x:

$$nx^{n-1} - x^n = 0$$

$$x^{n-1}(n - x) = 0$$

This gives two solutions $x = 0$ (which is not in the given range) and $x = n$.

So, the local maximum of $f(x)$ occurs at $x = n$, and the values ts:

$$f'(n) = \frac{n^x}{e^x}$$

Calculating $f(2x)$ and demonstrating the given inequality"

First, calculate $f(2x)$:

$$f(2x) = \frac{(2x)^n}{e^{2x}} = \frac{2^n x^n}{e^{2x}}$$

Now, let's demonstrate the given inequality:

$$\frac{2^n x^n}{e^x e^x} \leq \frac{2^n n^n}{e^n e^x}$$

Simplify the left side:

$$\frac{2^n x^n}{e^{2x}} \leq \frac{2^n n^n}{e^n e^x}$$

This inequality is consistent with the properties of the function $f(x)$ and the given range $x \geq n > 0$, where $f\,9x)$ is decreasing.

In summary, we have show that $f(x) = \frac{x^n}{e^x}$ decreases for $x \geq n > 0$, found the local maximum value of $f(x)$ at $x = n$, calculated $f(2x)$, and demonstrated the given inequality using the properties of the function and exponential growth.

Q. 3. Answer any three of the following: **4×3=12**

(a) Using Mean Value theorem, prove the inequality, $e^x \geq 1 + x$ for all $x \in R$.

Ans. The Mean Value Theorem (MVT) states that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In the case, let's consider the function $f(x) = e^x - (1 + x)$. We want to prove that $f(x) \geq 0$ for all $x \in R$.

1. First, notice that $f(0) = e^0 - (1 + 0) = 0$.
2. Next, calculate the derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}(e^x - (1 + x)) = e^x - 1$$

3. Now, let's usw the Mean Value Theorem on the interval $[0, x]$, where x is any real number greater than 0:

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{e^x - 1}{x}$$

where c is some point in $(0, x)$.

4. Rearrange the equation:

$$e^x - 1 = f'(c)\, x$$

5. Since e^x is positive for any real c, we have $e^x > 1$, which implies $e^x - 1 > 0$.

6. Therefore, we can concluide that f' (c) $x > 0$, since both f' (c) and x are positive.
7. Since f' (c) $x > 0$, we have $e^x - 1 > 0$, which means that f' $(x) = e^x - (1 + x) > 0$.

This prove thaty $e^x \geq 1 + x$ for all $x \in \mathbb{R}$ using the Mean Value Theorem.

(b) (i) The time in minutes, *t*, required for a rat to run through a maze depends on the number of trails, *n*, that the rat has pracitced.

$$t(n) = \frac{3n + 15}{n + 1}, n \geq 1$$

How does the change in *n* impact the change in *t*? Does there appear to be a limiting time in which the rat can complete the maze? How many trials are required so that the rat is able to finish the maze in under 5 minutes.

Ans. The given function is

$$t(n) = \frac{3n + 15}{n + 1}, \text{ where } n \geq 1.$$

The function $t(n)$ represents the time required for the rat to complete the maze based on the number of trials n practiced. To understand how the change in n impacts the change in t, we can analyze the behavior of the function. As n increases, the numerator $3n + 15$ becomes larger, which means the time t will increase. Similarly, as n increases, the denominator $n + 1$ also increases, but at a slower rate. This means that the fraction $t(n)$ will also increase as n increases. In other words, the more trials the rat practices, the longer it takes to complete the maze.

As n becomes very large, the $n + 1$ term in the denominator becomes negligible compared to the $3n + 15$ term in the numerator. This implies that the value of $t(n)$ will approach 3 as n gets larger and larger. In this sense, there seems to be a limiting time of 3 minutes for the rat to complete the maze. This limiting time is an asymptote that the function approaches as n grows.

We want to find the values of n for which $t(n) < 5$:

$$\frac{3n + 5}{n + 1} < 5$$

Cross-multiplying:

$$3n + 15 < 5n + 5$$

Simplifying:

$$2n > 10$$

Dividing both sides by 2:

$$n > 5$$

Since n must be an integer and $n \geq 1$ according to the domain given, the smallest integer greater than 5 is 6.

Therefore, the rat needs to practice at least 6 trials to finish the maze in under 5 minutes based on the given function.

Since n must be an integer and $n \geq 1$ according to the domain given, the smallest integer greater than 5 is 6. Therefore, the rat needs to practice at least 6 trials to finish the maze in under 5 minutes based on the given function.

In summary, the change in n directly affects the change in t, making the time increase as n increases. There appears to be a limiting time of 3 minutes as n becomes very large. The rat needs to practice at least 6 trials to finish the maze in under 5 minutes according to the given function.

(ii) National income in two economies X and Y is growing exponentially at 100 r_x% and 100 r_y % respectively (compounded continuously), where $r_x > r_y$. In year zero, national income was N^0_x in economy X and N^0_x in economy Y. If $N^0_x < N^0_{x}$, at what time will be national income become equal in both the economies?

Ans. If the national income in economies X and Y is growing exponentially at 100 r_x % and 100 r_y % respectively, compounded continuously, and $r_x > r_y$, then the growth rates can be expressed as decimal values r_x and r_y.

The growth formula for continuous compounding is given by:

$$A = P e^{rt}$$

where

- A is the final amount (national income in this case).
- P is the initial amount (national income at year zero).
- r is the growth rate (expressed as a decimal).
- t is the time in years.

For economy X:

$$A = N_{x0}\, e^{r_x t}$$

For economy Y:

$$A_y = N_{y0}\, e^{r_x t}$$

Given that $N_{x0} < N_{y0}$ we want to find the time t when the national income becomes equal in both economics. So, we set up an equation where $A_x = A_y$.

$$N_{x0}\, e^{r\,x\,t} = N_{y0}\, e^{r_x t}$$

Now, we can solve for t:

$$e^{rxt} = \frac{N_{y0}}{N_{xo}} e^{r\,x\,t}$$

Take the natural logarithim (ln) of both sides:

$$r_x t = \ln\left(\frac{N_{x0}}{N_{y0}}\right) + r_y\, t$$

Finally, solve for t:

$$t = \frac{\ln\left(\frac{N_{x0}}{N_{y0}}\right)}{r_x - r_y}$$

This formula gives the time it takes for the national income to become equal in both economies.

Since $r_x > r_y$ the term $r_x - r_y$ is positive, which means t will also be positive. The larger the difference between r_x and r_y, the longer it will take foer their national income to become equal.

(c) Use Newton's binomial formula to find the approximate value of $\sqrt{217}$, taking the degree of approximation as 2. Also find the upper bound on the absolute error.

Ans. Given $f(x) = \sqrt{x}$, we want to approximate $f(217)$. Choose a point near 217 foe which the square root is easily calculable. Let's choose $a = 225$, which gives $f(a) = \sqrt{225} = 15$.

Now, we can use the linear approximation formula.

$$L(x) = f(a) + f'(a)(x - a)$$

where:

- $L(x)$ is the linear approximation of $f(x)$ near a.
- $f(a)$ is the value of $f(x)$ at a.
- $f'(a)$ is the derivative of $f(x)$ evaluated at a.
- x is the point at which we want to approximate $f(x)$.

For $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$. Evaluating $f'(a) = 225$.

$$f'(225) = \frac{1}{2\sqrt{225}} = \frac{1}{30}$$

Plugging in the values into the linear approximation formula:

$$L(217) = f(225) + f'(225)(217 - 225)$$

$$L(217) = 15 + \frac{1}{30} \cdot (-8) = 14.7333$$

So, using linear approximation, the approximation values of $\sqrt{217}$ is approximately 14.7333.

To find the upper bourd on the absolute error, we can use the error formula for linear approximation.

$$E \leq \frac{M}{2} \cdot |x - a|$$

Where M is the maximum value of the second derivative of $f(x)$ in the interval between a and x. For $f(x) = \sqrt{x}$, the second derivative $f''(x) = -\frac{1}{4x^{3/2}}$, which is negative and decreasing. So, M is maximum at $a = 225$.

$$M = |f''(225)| = \frac{1}{4.225^{3/2}} \quad \frac{1}{27000}$$

Plugging in the values into the error formula:

$$E \leq \frac{1}{2.27000} \cdot |217 - 225| = \frac{8}{540000} \approx 0.0001481$$

The upper bound on the absolute error is approximately 0.0001481 which means that the actual value of the given square root is withing this range of the linear approximation value 14.7333.

(d) An investment project incurs an initial loss of C_0. Thereafter it does not incur any losses and the sum of later profits is greater than the initial loss. Show that the project has a unique positive internal rate of return.

Ans. The internal rate of return (IRR) is the discount rate that makes the net present value (NPV) of a project zero. In other words, it's the interest rate at which the present value of future cash flows equals the initial investment.

Given that the investment project incurs an initial loss of C_0 and later profits are greater than the initial loss, we can define the cash flows as follows:

- Initial investment – C_0
- Later profit $P_1, P_2, ..., P_n$ where $P_i > C_0$ for all i.

The NPV of the project can be calculated as:

$$\text{NPV} = -C_0 + \frac{P_1}{(1+\text{IRR})^1} + \frac{P_2}{(1+\text{IRR})^2} + ... + \frac{P_n}{(1+\text{IRR})^n}$$

Since, $P_i > C_0$ for all i, each term in the NPV equation has a positive numerator (profit) and a denominator greater than 1 (due to the positive IRR), making the overall contribution to NPV positive. Therefore, it's clear that the NPV is positive for any positive IRR value.

Now, let's consider the limiting case where the IRR approaches 0. As the IRR gets smaller, the denominators of the fractions in the NPV equation become larger. Since all the P_i terms are positive, the sum of these fractions will become larger in magnitude as the IRR approaches 0, making the NPV negative. This is because the project is incurring an initial loss.

On the other hand, as the IRR becomes larger, the denominators become smaller, causing the NPV to increase since the profits are larger than the initial loss. Therefore, the NPV starts positive at IRR = 0 and keeps increasing as the IRR increases.

Since the NPV function is continuous and changes from negative to positive as the IRR increases, there must be a unique positive IRR that makes the NPV equal to zero. This is because the NPV starts negative, crosses zero, and then remains positive for larger IRR values.In conclusion, based on the fact that the investment project incurs an initial loss followed by profits and that the profits are greater than the initial loss, the project has a unique positive internal rate of return. This is because the NPV changes from negative to positive as the IRR increases, indicating the existence of a single IRR value that makes the NPV zero.

Q. 4. Answer any three of the following **4×3=12**

(a) If f is a continuous function on the interval [0, 1] with f (o) > 0 and f (1) < 1, then there is some number c Ñ (0, 1) which satisfies f (c) = c.

Ans. The statement is a direct application of the intermediat values Theorem, where is a fundamental theorem in calculate that applies to continuous functions.

The intermediate Value Theorem states that if a continuous function $f(x)$ is defined on a closed interval $[a, b]$ and y is any value between $f(a)$ and $f(b)$ (including the values $f(a)$ and $f(b)$ themselves), then there exists at least one value c in the interval $[a, b]$ such that $f(c) = y$.

In this given problem, we have $a = a$, $b = 1$, $f(0) > 0$, and $f(1) < 1$. According to the Intermediate Value Theorem, since $f(0)$ is positive and $f(1)$ is less than 1, there must be some value c in the interval [0, 1] such that $f(c) = c$, because c will be a values between $f(0)$ and $f(1)$.

So, in this case, the intermediate Value Theorem guarantees the existence of a number c in the interval (0, 1) such that $f(c) = c$. This theorem is a consequences of the continuity of the function $f(x)$ on the interval [0, 1] and the specific conditions goven in the problem.

(b) Suppose that the value of win $W(t)$ is given as the following function of time: $W(t) = 1000 \,.\, e^{\sqrt{\frac{1}{4}}}$ in crores of rupees ($t = 0$ denotes the present). At an interest rate 10% compounded continuously and assuming zero storage costs, what is the optimal time to sell the wine? Interpret the first order condition.

Ans. To find the optimal time to sell the wine, we need to maximize the present value of the wine's value $W(t)$ at the given interest rate of 10% compounded continuously. We'll consider the time t as the time at which the wine is sold.

The present value PV of a future amount F at time t using continuous compounding is given by

$$PV = F \,.\, e^{-rt}$$

where:

- F is the future amount (in this case, the value of the wine at time t).
- r is the interest rate (as a decimal)
- t is the time in years.

Given that the value of wine $W(t)$ os $1000 \,.\, e^{\sqrt{1/4}}$ crore rupees, we can use this formula to calculate the present value of the wine's value at time t:

$$PV(t) = 1000 - e^{\sqrt{1/4}} \cdot e^{-0.10\,t}$$

Simplify the expression:

$$PV(t) = 1000 - e^{(\sqrt{t/4} - 0.10\,t}$$

Now, to find the optimal time to sell the wine, we need to find the time t that aximizes the present value $PV(t)$. We'll takes the derivatives of $PV(t)$ with espect to t and set it equal to zero to find the critical points.

$$\frac{d}{dt} PV(t) = 1000 - \left(\frac{1}{4.2 \cdot \sqrt{t}} - 0.10\right) \cdot e^{(\sqrt{t}/4 - 0.10\,t)} = 0$$

Solving for t:

$$\frac{1}{4.2\sqrt{t}} - 0.10 = 0$$

Simplify:

$$\frac{1}{4.2\sqrt{t}} = 0.10$$

Solve for t:

$$t = \left(\frac{1}{4.2.0.10}\right)^2 = \frac{1}{0.0006} = 1250$$

So, the optimal time to sell the wine is approximately $t = 1250$ years.

Interpreting the first-order condition:

The first-order condition in optimization is the condition where the derivative f the objective function is set equal to zero. In this case, the first-order condition is:

$$\frac{1}{4.2\sqrt{t}} - 0.10 = 0$$

This condition implies that the derivative of the present values of the wine's alue with respect to time is zero at the optimal time t. In other words, the ptimal time to sell the wine is the time when the rate of change of its present alue with respect to time is zero, indicating that the present value is neither ncreasing nor decreasing at that point. This is the point where the value of elling the wine is maximizedf under the given conditions.

(c) Let $f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, does the function f have an inverse function g? If yes, find the inverse and $g'\left(\frac{1}{2}\ln 3\right)$.

Ans. To determine if the function $f(x) = \dfrac{\ln\left(\frac{1+x}{1-x}\right)}{2}$ has an inverse function , we need to check if the function is one-to-one (injective) on its domain. A unction has an inverse if and only it if is bijective, meaning it is both injective nd surjective.

First, let's find the domain of $f(x)$. The natural logarithm ln is defined only r positive arguments, so $\frac{1+x}{1-x} > 0$. Solving this inequality ofr x gives us -1

$< x < 1$. Thus, the domain of $f(x)$ is $-1 < x < 1$.

Now, let's analyze the derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}\left(\frac{\ln\left(\frac{1+x}{1-x}\right)}{2}\right)$$

Using the chain rule and the logarithmic differentiation rules, we can claculate the derivative as follows:

$$f'(x) = \frac{1}{2}\cdot\frac{1}{\frac{1+x}{1-x}}\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$f'(x) = \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{(1-x)-(1+x)}{(1-x)^2}$$

$$f'(x) = -\frac{1}{(1-x)^2}$$

The derivative is always negative in the domain $-1 < x < 1$, meaning the function is strictly decreasing within this interval. This implies that the function is injective (one-to-one) in its domain.

Therefore, since $f(x)$ is injective within its domain, it has an inverse function g.

To find the inverse function g, we switch the roles of x and y and solve for y in the equation

$$y = \frac{\ln\left(\frac{1+x}{1-x}\right)}{2}$$

$$x = \frac{\ln\left(\frac{1+x}{1-x}\right)}{2}$$

$$x = \frac{\ln\left(\frac{1+x}{1-x}\right)}{2}$$

Now, we solve for y:

$$2x = \ln\left(\frac{1+y}{1-y}\right)$$

$$e^{2x} = \frac{1+y}{1-y}$$

$$(1-y)\,e^{2x} = 1+y$$

$$(1-y)\,e^{2x} - 1 = y$$

$$e^{2x} - e^{2x}y - 1 = y$$

$$e^{2x} - 1 = y + e^{2x}y$$

$$y = \frac{e^{2x} - 1}{1 + e^{2x}}$$

So, the inverse function $g(x)$ is:

$$g(x) = \frac{e^{2x} - 1}{1 + e^{2x}}$$

Finally, let's find $g'\left(\frac{1}{2}\ln 3\right)$:

$$g'(x) = \frac{d}{dx}\left(\frac{e^{2x} - 1}{1 + e^{2x}}\right)$$

Using the quotient rule, the derivative can be calculated as follows:

$$g'(x) = \frac{(1 + e^{2x})(2e^{2x}) - (e^{2x} - 1)(2e^{2x})}{(1 + e^{2x})^2}$$

Plugging in $x = \frac{1}{2}\ln 3$:

$$g'\left(\frac{1}{2}\ln 3\right) = \frac{(1 + 3)(2\sqrt{3}) - (\sqrt{3} - 1)(2\sqrt{3})}{(1 + 3)^2}$$

$$g'\left(\frac{1}{2}\ln 3\right) = \frac{8\sqrt{3} - 2\sqrt{3}}{16} = \frac{6\sqrt{3}}{16} = \frac{3\sqrt{3}}{8}$$

So, $g'\left(\frac{1}{2}\ln 3\right) = \frac{3\sqrt{3}}{8}$.

(d) Given f and g are not differentiable functions, show $g(x) = f(ax + b)$ (where a and b are real numbers) is convex if f is conves.

Ans. To shoa that $g(x) = f(ax + b)$ is convex if f is convex, we can use the definition of convexity and Jensen's inequality.

A function f is convex on an interval I if, for all x_1, x_2 in I and for all t in the interval [0, 1],

$$f(t x_1 + (1\text{-}t) x_2) \leq t f(x_1) + (1 - t) f(x_2)$$

Now let's prove that $g(x)$ is convex

Given that f is convex, we want to show that $g(x) = f(ax + b)$ satisfies the convexity property.

Let x_1 and x_2 be two arbitary points in the domanin of $g(x)$, and let t be a value in the interval [0, 1]. We need to prove that:

$$g(t x_1 + (1 - t) x_2) \leq t g(x_1) + (1 + t) g(x_2)$$

Substitute $g(x) = f(ax + b)$:

$$f(a(tx_1 + (1 - t) x_2) + b) \leq t f(ax_1 + b) + (1 - t) f(ax_2 + b)$$

Now, notice that $a(t x_1 + (1 - t) x_2) + b$ is the linear combination $ax_1 + (1 - t)$

ax_2 + ??? which imples that it lies between $ax_1 + b$ and $ax_2 + b$ on the real line Because f is convex, we can apply the convexity property.

$$f(ax_1 + b) \leq t f(ax_1 + b) + (1 - t) f(ax_2 + b)$$
$$f(ax_1 + b) \leq t f(ax_1 + b) + (1 - t) f(ax_2 + b)$$

Adding the above two inequanlities gives us:

$f(ax_1 + b) + f(ax_2 + b) \leq t f(ax_1 + b) + (1 - t) f(ax_2 + b) + t f(ax_1 + b) + (1 - t) f(ax_2 + b)$

$f(ax_1 + b) + f(ax_2 + b) \leq 2t f(ax_1 + b) + 2(1 - t) f(ax_2 + b)$

Divided both sides by 2:

$$\frac{1}{2}(f(ax_1 + b) + f(ax_2 + b)) \leq t f(ax_1 + b) + (1 - t) f(ax_2 + b)$$

Since the left=hand side is f evaluated at a convex combination of $ax_1 + b$ and $ax_2 + b$, this shows that $g(x) = f(ax + b)$ is convex.

Therefore, if f is a convex function, then $g(x) = f(ax + b)$ is also convex for any real numbers a and b.

Q. 5. Answer any three of the following: 3×5=15

(a) The graph of the equation $x^2 y - 3y^3 = 2x$ passes through the point $(x, y) = (-1, 1)$. Find the slope of the graph at this point. Find the points where function is not differentiable. Does the curve have horizontal tangent?

Ans. To find the slope of the graph at the point (– 1, 1), we need to calculate the derivation of the given equation with respect to x, and then substitute $x = -1$ and $y = 1$ into the derivation.

The equations is:

$$x^2 y - 3y^2 = 2x$$

Let's first implicitly differentiate the equation with respect to x:

$$\frac{d}{dx}(x^2 y - 3y^2) = \frac{d}{dx}(2x)$$

$$2xy + x^2 \frac{dy}{dx} - 9y^2 \frac{dy}{dx} = 2$$

Now, plug in $x = -1$ and $y = 1$:

$$2(-1)(1) + (-1)^2 \frac{dy}{dx} - 9(1)^2 \frac{dy}{dx} = 2$$

$$-2 + \frac{dy}{dx} - 9\frac{dy}{dx} = 2$$

$$-11\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = -\frac{4}{11}$$

So, the slope of the graph at the point (– 1, 1) is $- -\frac{4}{11}$.

To find the points where the function is not differentible, we need to look for points where the derivative does not exist or is underfined. In this case, the derivative exists for the entire domain of the equation. However, if the equation is not satisfied at a certain point, then that point wouldn't be differentiable for the curve described by the equation.

Finaally, to determine if the curve has a horizontal tangent at any point, we need to find points where the derivative is zero. In this case, we've already found that the derivative at (– 1, 1) is $-\frac{4}{11}$, which is not zero. Therefore, the curve does not have a horizontal tangent at the point (– 1, 1), and the slope is not zero at this point.

(b) (i) Find the limit: $\lim_{x \to \infty} \left(\frac{2 + 3x^m}{1 - x^n} \right) m, n \in N.$ 2

Ans. To find the limit of the expression $\lim_{x \to \infty} \frac{2 + 3x^n}{1 - x^n}$, where m and n are natural numbers, we can analyze the behaviout of the expression as x approaches infinity.

Case 1: $m > n$

If $m > n$, as x goes to infinity, the term $3x^n$ will dominate the behaviour of the numerator since its degree is higher than that of the denominator $1 - x^n$. As a result, the expression will approach infinity:

$$\lim_{x \to \infty} \frac{2 + 3x^n}{1 - x^n} = \infty$$

Case 2: $m = n$

If $m = n$, the degree of the numerator and denominator are the same. The limit becomes an indeterminate from $\left(\frac{\infty}{\infty} \right)$. We can apply L' Hopital's rule to find the limit.

$$\lim_{x \to \infty} \frac{2 + 3x^n}{1 - x^n} = \lim_{x \to \infty} \frac{2m\, x^{m-1}}{-nx^{m-1}}$$

So, when $m = n$, the limit approaches 0.

Case 3: $m < n$

If m n, the degree of the denominator $1 - x^n$ will dominate the behaviour of the fraction as x goes to infinity. In this case, the fraction will approach 0:

$$\lim_{x \to \infty} \frac{2 + 3x^n}{1 - x^n} = 0$$

To summarize:

- If $m > n$, the limit is ∞.

- If $m = n$, the limit is 0.
- if $m < n$, the limit is 0.

The behaviour of the limit depends on the relationship between the exponents m and n.

(ii) Let $f(x)$ $\dfrac{\log\left(1+\dfrac{x}{p}\right)-\log\left(1-\dfrac{x}{q}\right)}{q}$, where p and q are positive constants. Can you define $f(x)$ at $x = 0$ so as to make the function continuous at $x = 0$?

Ans. To make the function $f(x) = \dfrac{\log\left(1+\dfrac{x}{p}\right)-\log\left(1-\dfrac{x}{p}\right)}{z}$ continuous at $x = 0$, we need to find the value of $f(0)$ that makes the limit of $f(x)$ as x approaches 0 well-defined.

First, let's analyze the function $f(x)$ as x approaches 0:

$$f(x) = \frac{\log\left(1+\dfrac{x}{p}\right)-lof\left(1-\dfrac{x}{p}\right)}{z}$$

As x approaches 0, both terms inside the logarithms log $\left(1+\dfrac{x}{p}\right)$ and log $\left(1-\dfrac{x}{q}\right)$ approach 0, and the denominator x approaches 0 as well. Therefore, the entire expression $f(x)$ becomes an indeterminate form $\dfrac{0}{0}$.

To make $f(x)$ continuous at $x = 0$, we need to find the limit of $f(x)$ as x approaches 0 and determine the value of $f(0)$ that corresponds to that limit.

Let's compute the limit:

$$\lim_{x \to 0} \frac{\dfrac{1}{p+x}+\dfrac{1}{q+x}}{1} = \lim_{x \to 0} \frac{\log\left(1+\dfrac{x}{p}\right)-\log\left(1-\dfrac{x}{q}\right)}{z}$$

Apply L'Hopital's rule, as the limit is of the indeterminate form $\dfrac{0}{0}$.

$$\lim_{x \to 0} \frac{\dfrac{1}{p+x}+\dfrac{1}{q+x}}{1} = \frac{1}{p}+\frac{1}{q}$$

So, the limit as x approaches 0 is $\dfrac{1}{p}+\dfrac{1}{q}$

To make $f(x)$ continuous at $x = 0$, we set $f(0)$ equal to the limit we found:

$$f(0) = \frac{1}{p} + \frac{1}{q}$$

By setting $f(0)$ to $\frac{1}{p} + \frac{1}{q}$, we ensure that the function $f(x)$ is continuous at $x = 0$. The means that the function's behaviour at $z = 0$ is compatible with its behaviour as x approaches 0, and there are no discontinuities at that point.

(c) (i) Consider two cash flows, *A* and *b*. For cash flow *A*, you receive ₹ 10 every year for 5 years with the first payment being a year from now. For cah flow *B*, you receive ₹ *x* every year forever with the first payment being today. What is the value of *x* so that cash flow *B* has the same present value as cash flow *A*, given that the rate of interest is 6% per annum (compounded annually)? **3**

Ans. To find the value of x such that cash flow B has the same present value as cash flow A, we need to equate their present values and solve for x.

Cash flow A is a series of payments of ` 10 every years for 5 years, with the first payment being received a year from now. The formula to calculate the present value of a series of payments is:

$$PV_A = \frac{C}{(1+r)^n} + \frac{C}{(1+r)^{n+1}} + ... + \frac{C}{(1+r)^{n+k-1}}$$

Where:

- PV_A is the present value of cah flow A.
- C is the cash payment (` 10 per year).
- r is the interest rate (6% per annum = 0.06).
- n is the number of years unitil the first payment (1 year).
- k is the number of payments (5 payments).

Substituting the values:

$$PV_A = \frac{10}{(1+0.06)^1} + \frac{10}{(1+0.06)^2} + ... + \frac{10}{(1+0.06)^3}$$

Similarly, for cash flow B, the formula for the present value of a perpetuity is:

$$PV_B = \frac{x}{r}$$

Where:

- PV_B is the present value of cash flow B.
- x is the cash payment per year (to be determined).
- r is the interest rate (6% per annum = 0.06)

Now, equate PV_A and PV_B and solve for x:

$$\frac{10}{(1+0.06)^1} + \frac{10}{(1+0.06)^2} + ... + \frac{10}{(1+0.06)^3} = \frac{x}{0.06}$$

Calculate the left side of the equation:

$$\frac{10}{1.06}+\frac{10}{1.06^2}+\frac{10}{1.06^3}+\frac{10}{1.06^4}+\frac{10}{1.06^5}\approx 39.5782$$

Now solve for x:

$$\frac{x}{0.06} = 39.5782$$

$$x = 0.06 \times 39.5782$$

$$x \text{ K } 2.3747$$

So, in order for cash flow B to have the same present value as cash flow A, the value of x should be approximately ₹ 2.3747.

(ii) If $f(x)$ and $g(x)$ are differentiable functions of x, express the elasticity of $h(x) = e^{f(x)\,g(x)}$ w.r.t. x in terms of $E_x f$ and $E_x g$ which are the elasticities of $f(x)$ and $g(x)$ w.r.t. x respectively. **2**

Ans. To express the elasticity of the function $h(x) = e^{f(x)\,g(x)}$ with respect to x in terms of the elasticities E_{xF} and E_{xg} of $f(x)$ and $g(x)$ respectively, we'll follow a similar approach as before.

The elasticity of a function y with respect to a variable x is given by:

$$E_x = \frac{\frac{dx}{dt} - x}{y}$$

Now, for the function $h(x) = e^{f(x)\,g(x)}$. Let's find the derivation $\frac{dh}{dx}$.

$$\frac{dh}{dx} = \frac{d}{dx}(e^{f(x)\,g(x)})$$

Using the chain rule:

$$\frac{dh}{dx} = e^{f(x)\,g(xi)} \cdot \frac{d}{dx}(f(x)\,g(x))$$

Now, we have the numerator of the elasticity formula, $\frac{dh}{dx} \cdot x$, which as:

$$\frac{dh}{dx}\cdot x = x \,.\, e^{f(x)\,g(x)} \,.\, \left(\frac{df}{dx}\cdot g(x) + f(x)\cdot\frac{dg}{dx}\right)$$

And the denominator, $h(x)$, is:

$$h(x) = e^{f(x)\,g(x)}$$

Now, we can express the elasticity E_h as:

$$E_h = \frac{\frac{dh}{dx}\cdot x}{h(x)} = \frac{x \,.\, e^{f(x)\,g(x)} \,.\left(\frac{df}{dx}\cdot g(x) + f\,()\cdot\frac{dx}{dx}\right)}{e^{f(x)\,g(x)}}$$

Simplifying the expression further:

$$E_h = x \,.\left(\frac{df}{dx}\cdot g(x) + f(x)\cdot\frac{dg}{dx}\right)$$

Now, we can substitute E_{xf} and E_{xg} to represent the elasticities of $f(x)$ and $g(x)$ respectively:

$$E_h = x \, . \, (E_{xf} \, . \, g(x) + f(x) \, . \, E_{xg})$$

This is the expression for the elasticity E_h of the function $h(x) = e^{f(x)\, g(x)}$ with respect to x, in terms of the elasticities E_{zf} and E_{xg} of $f(x)$ and $g(x)$ respectively.

(d) $Q^d = f(P + t)$ and $Q^s = g(P)$ where f and g are differenitable functions with $f' < 0$ and $g' > 0$. Use the equilibrium condition $Q^d = Q^s$ to find an expression for $\frac{dP}{dt}$. Also comment on its sign. Find the expression for $\frac{d(P+t)}{dt}$ and find its range.

Ans. Given the equilibrium condition $Q^t = Q^s$, where Q^d is the quantity demand and Q^s the quantity supplied, expressed in terms of the price P and a paramenter t, we have:

$$f(P+t) = g(P)$$

Now, let's differentiate both sides of this equation with respect to t using the chain rule:

$$\frac{d}{dt}[f(P+t)] = \frac{d}{dt}[g(P)]$$

The left side:

Using the chain rule, we have:

$$\frac{d}{dt}[f(P+t)] = f'(P+t) \cdot \frac{d}{dt}(P+t)$$

And the right side:

Since $g(P)$ is a function of P, its derivatives with respect to t is zero.

So, we have

$$f'(P+t) \cdot \frac{d}{dt}(P+t) = 0$$

Now, let's solve for $\frac{d}{dt}(P+t)$:

$$\frac{d}{dt}(P+t) = 0$$

This implies that the rate of change of $P + t$ with respect to t is zero. In other words, the equilibrium price $P + t$ doesn't change with time. This is resonable, as $P + t$ represents the equilibrium proice, and if the equilibrium is maintained, the price won't change over time.

Now, let's consider the expression for $\frac{dp}{dt}$, which represents the rate of change of the equilibrium price P with respect to time t:

$$\frac{dp}{dt} = \frac{d}{dt}P$$

This implies that the rate of change of the equilibrium price P with respect to time is the same as the rate of change of $P + t$, which we already established to be zero. Therefore, $\frac{dp}{dt} = 0$, indicating that the equilibrium price P doesn't change over time.

In summary:

1. The equilibrium condition $Q^d = Q^s$ leads to this conclusion that the equilibrium perice P doesn't change with time $\left(\frac{dp}{dt} = 0\right)$.

2. The rate of change of $P + t$ with respect to time is $\frac{d}{dt}(P+t) = 0$, indicating that the equilibrium perice $P + t$ doesn't change with time either.

Q. 6. Answer any two of the following: 6×2=12

(a) The monopolist with the cost function $C(x) = \frac{1}{2}x^2$, with quantity x, faces a demand curve x m= $12 - p$, where p is the price.

(i) Find equilibrium price and quantity.

Ans. To find the equilibrium price and quanity under monopoly, we need to equate the monopolist's marginal cost)MC) to the marginal revenue (MR) derived from the demand curve.

Given the cost function $C(x) = \frac{1}{2}x^2$ and the demand curve $a = 12 - p$, we need to find the monopolist's profit-maximizing quantity x^* and then calculate the corresponding price p^*.

Marginal cost (MR) is the derivative of the cost function.

$$MC = \frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$$

Marginal revenue (MR) can be obtained by deriving the total revenue function and then finding its derivative.

$$TR = p \cdot x = p \cdot (12 - p) = 12p - p^2$$

$$MR = \frac{dTR}{dx} = 12 - 2p$$

To find the profit-maximizing quantity x^*, we ser MR equal to MC:

$$12 - 2P^* = x^*$$

Substitute the demand curve $x = 12 - p$ into the equation.

$$12 - 2p^* = 12 - p^*$$

$$p^* = 4$$

Now that we have the equilibrium price p^*, we cwn dind the corresponding equilibrium quantity x^* using the demand curve:

$$x^* = 12 - p^* = 12 - 4 = 8$$

So, under monopoly, the equilibrium price is \$4 and the equilibrium quantity is 8 units.

(ii) What would be the equilibrium price and quantity if the monopolist is forced to take the price as given as under perfect competition? Compare the profits under monopoly and perfect competition.

Ans. Under perfect competition, the monopolist would be a price taker, meaning they have to take the market price as given. In this case, the demand curve effectively becomes $x = 12 - p$ (same as before)

Equilibrium quantity (x^*) under perfect competition is determined by setting the demand equal to the supply, which is $x^* = 12 - p$. Substituting $p = 4$ from the pervious part:

$$x^* = 12 - 4 = 8$$

Again, the equilibrium quantity is 8 units. Howerver, since the monopolist is forced to take the market price as given, the equilibrium price is not determined by the monoplist but is the result of the market interaction, which is still \$4.

(iii) To ensure that the monopolist acts like a perfectly competitive firm, a specific tax oft per unit is imposed on him. Find the equilibrium output, t and show that it is actually negative, What does it imply?

Ans. To make the monopolist behave like a perfectly competitive firm, a specific tax t per unit is imposed on them. The monopolist's effective cost becomes $C(x) + t = \frac{1}{2}x^2 + t.$

Now, the monopolist's profit-maximizing condition becomes:

$$12 - 2p = x^*$$

$$12 - 2p = \frac{1}{2}x^2 + t$$

Substitute the demand curve $x = 12 - p$ into the second equations:

$$12 - 2p = \frac{1}{2}(12 - p)^2 + t$$

Solve for t:

$$t = 12 - 2p - \frac{1}{2}(12 - p)^2$$

Substitution the equilibirum price $p^* = 4$:

$$t = 12 - 2 . 4 - \frac{1}{2}(12 - 4)^2 = -2$$

So, the specific tax t is $-$ 2. This implies that the monopolist is actually receiving a subsidy of 2 units per unit of output, which encourages them to produces more. This scenario doesn't represent a real-world taxation situation but rather a mathematical scenario that leads to the monopolist producing more than theu would under normal conditions.

(b) (i) Let the function $f(x) = (6 - x^2)\sqrt{x^2 - 4}$ be defined over [– 6, – 2], Find the extreme points of f.

Ans. The critical points opccur where the derivative of the function is zero or undefined. let's find the derivative of $f(x)$:

$$f(x) = (6 - x^2)\sqrt{x^2 - 4}$$

Using the product rule:

$$f'(x) = (6-x^2).\frac{d}{dx}\left(\sqrt{x^2-4}\right)+\left(\sqrt{x^2-4}\right)\cdot\frac{d}{dx}(6-x^2)$$

Simplify the derivatives:

$$f'(x) = (6-x^2).\frac{x}{\sqrt{x^2-4}}+\left(\sqrt{x^2-4}\right)\cdot(-2x)$$

Setting $f'(x)$ equal to zero and solving for x:

$$(6-x^2).\frac{x}{\sqrt{x^2-4}}+\left(\sqrt{x^2-4}\right)\cdot(-2x) = 0$$

Simplify:

$$(6-x^2).x - 2x(\sqrt{x^2-4}) = 0$$

$$(6x - x^3) - 2x(\sqrt{x^2-4}) = 0$$

$$x(6 - x^2 - 2\sqrt{x^2-4}) = 0$$

This gives three possible critical points: $x = 0$, $x = 2$, and $x = -2$.

Step 2: **Analyze the Critical Points**

We'll analyze the behaviour of the function at each of the critical points.

- At $x = 0$.

  $f(0) = (6 - 0^2)\sqrt{o^2 - 4} = 0.$

- At $x = 2$:

  $f(2) = (6 - 2^2)\sqrt{2^2 - 4} = 0.$

- At $x = -2$:

  $f(-2) = (6 - (-2)^2)\sqrt{(-2)^2 - 4} = 0.$

Summary: The function $f(x) = (6 - x^2)\sqrt{x^2 - 4}$ defined over the interval [– 6, – 2] has extreme points at $x = 0$, $x = 2$ and $x = -2$, where the function evaluates to zero. These points represetn points where the function reaches an exterme value within the given interval. Note that these points are not true local maxima or minima in the classical sense but are points of inflection where the behaviour of the function changes.

(ii) Determine the concavity/convexity of the following function $f(x) = (e^{2x} + 4e^{-x})^2$.

Ans. To determine the concavit/convexity of the function $f(x) = (e^{2x} + 4 e{-}x)^2$, we need to analyze the second derivative of the function. The second derivative test will help us determine whether the function is convave up (convex) or convace down (convave) at different points.

Let's start by finding the first and second derivatives of $f(x)$:

Given: $$f(x) = (e^{2x} + 4e^{-x})^2$$

First Derivative ($f'(x)$):

Using the chain rule, er have:

$$f'(x) = 2(e^{2x} + 4e^{-x})(2e^{2x} - 4e^{-x})$$

Simplify the expression:

$$f'(x) = 4(e^{4x} - e^{-x})$$

Second Derivative ($f''(x)$):

Using the chain rule again, we find the second derivative:

$$f'''(x) = 4(e^{4x} - e^{-x})$$

Now, let's analyze the sign of the second derivative to determine the concavity/convexity:

1. If $f''(x) > 0$ for all x in the domain, the function is convave up (convex).
2. If $f''(x) < 0$ for all x in the doman, the fuction is convave down (concave).
3. If $f''(x)$ changes sign, the convavity/convexity changes at that point.

Let's evaluate $f''(x)$:

$$f''(x) = 4(4e^{4x} + e^{-x})$$

Since e^{4x} and e^{-x} are always positive, and the constant factors are positive, the entire expression $f''(x)$ is always positive. This means that $f(x)$ is concave up (convex) for all value of x in its domain.

So , the function $f(x)$ is convex for all values of x in its domain.

(c) Let $f(x) = x - 2\ln(x+1)$

(i) Determine where $f(x)$ is increasing/ decreasing.

Ans. To determine where the function $f(x) = x - 2\ln(x+1)$ is increasing or decreasing, we need to analyze its first derivative. $f'(x)$, and its sign.

First, find the first derivative $f'(x)$:

$$f(x) = x - 2\ln(x+1)$$

Differentiate both sides with respect to x:

$$f'(x) = 1 - \frac{2}{x+1}$$

To find where $f(x)$ is increasing or decreasing, we examine the sign of $f'(x)$.

- If $f'(x) > 0$, then $f(x)$ is increasing
- If $f'(x) < 0$, then $f(x)$ is decreasing.

Now, let's analyze $f'(x)$:

$$f'(x) = 1 - \frac{2}{x+1}$$

Set $f'(x) > 0$

$$1 - \frac{2}{x+1} > 0$$

$$1 > \frac{2}{x+1}$$

$$x + 1 > 2$$

$$x > 1$$

Set $f'(x) < 0$

$$1 - \frac{2}{x+1} < 0$$

$$1 < \frac{2}{x+1}$$

$$x + 1 < 2$$

$$x < 1$$

So, the function $f(x)$ is increasing for $x > 1$ and decreasing for $x < 1$.

(ii) Find possible extreme points and inflexion points. Does the function have global maximum/minimum point(s)?

Ans. To find possible extreme points and inflection points, we need to analyze the second derivative $f''(x)$ and its sign.

Find the second derivative $f''(x)$ by differentiating $f'(x)$.

$$f'(x) = 1 - \frac{2}{x+1}$$

$$f''(x) = \frac{2}{(x+1)^2}$$

An inflection point occurs where the sign of the second derivative changes. Since $f''(x)$ is always positive (as the denominator is positive), there are no inflection points for this function.

To find possible extreme points, set the first derivative $f'(x)$ equal to zero.

$$1 - \frac{2}{x+1} = 0$$

$$\frac{2}{x+1} = 1$$

$$x + 1 = 2$$

$$x = 1$$

So, the possible extreme point is at $x = 1$. To determine whether this point is a maximum or minimum, we can examins the sign of $f'(x)$ around $x = 1$:

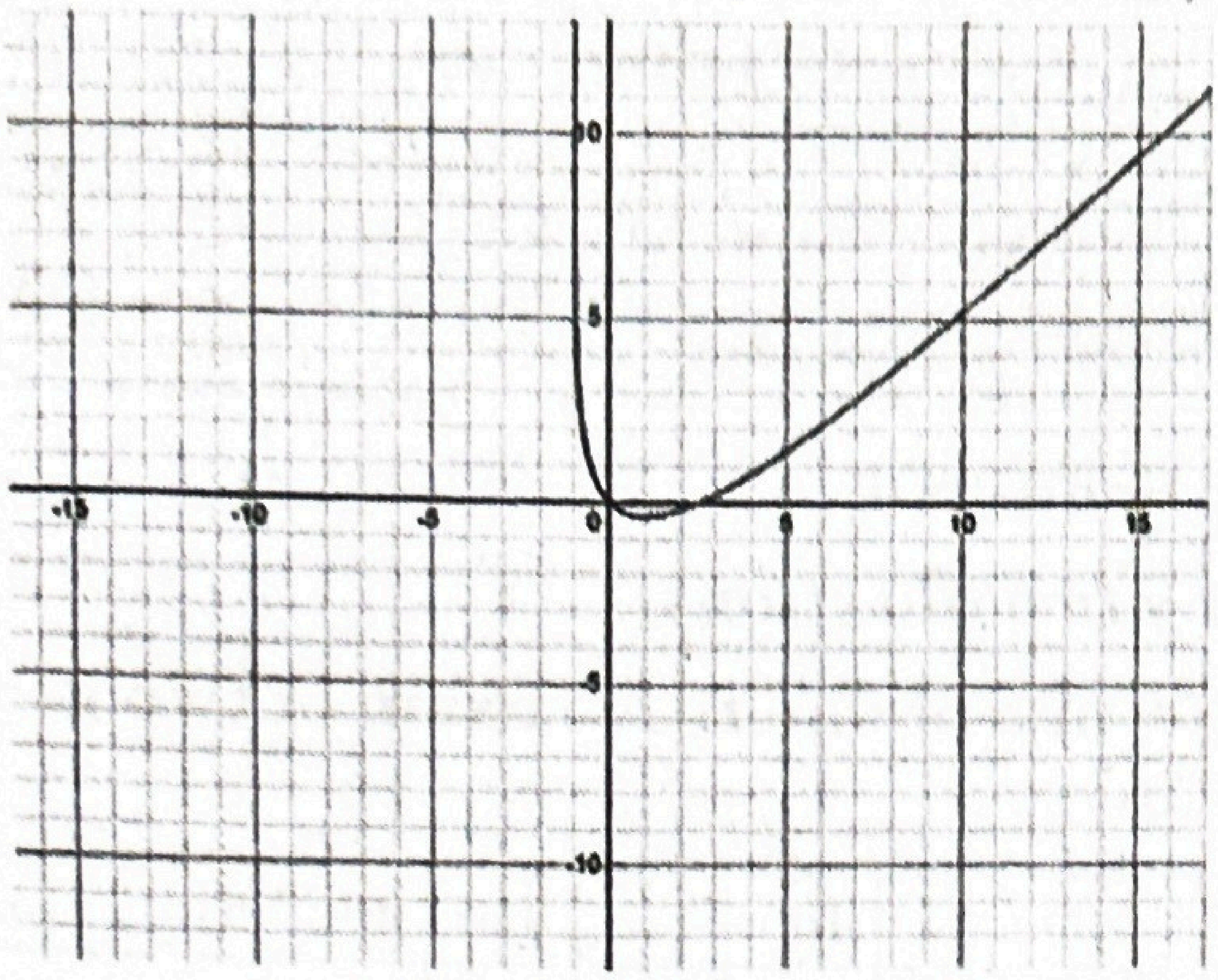
- $x < 1$: $f'(x) < 0$ (decreasing)
- $x > 1$: $f'(x) > 0$ (increasing)

Since the function changes from decreasing to increasing, the point at $x = 1$ is a local minimum.

In summary, the function $f(x) = x - 2 \ln(x + 1)$ is increasing for $x > 1$ and decreasing for $x < 1$. It has a local minimum at $x = 1$ and no inflection points. It does not have gobal maximum or minimum points, as it is unbounded on both ends.

(iii) Sketch the graph of $f(x)$.

Ans.



PwD **(iv) Determine the intervals of concavity/convexity of the function $g(x) = x^4 - 12x^2$.**

Ans. To determine the intervals of convavity and convexity of the function $g(x) = x^4 - 12x^3$, we need to anlayze the second derivative $g''(x)$ and its sign. The sign of the second derivative will help us identify where the function is concave up (convex) or concave down (concave).

Let's start by finding the second derivative $g''(x)$ of $g(x)$:

Given: $$g(x) = x^4 - 12x^2$$

Second derivative ($g''(x)$):

Differentiate $g(x)$ with respect to x to find the first derivative $g'(x)$:

$$g'(x) = 12x^2 - 24x$$

Then, differentiate $g'(x)$ with respect to x to find the second derivative $g''(x)$

$$g''(x) = 12x^2 - 24$$

Now, let's analyze the sign of $g''(x)$:

- If $g''(x) > 0$, then $g(x)$ is concave up (convex).
- If $g''(x) < 0$, then $g(x)$ is concave down (concave).

Solve $g''(x) > 0$ to find intervals of convavity.

$$12x^2 - 24 > 0$$
$$12x^2 > 24$$
$$x^2 > 2$$
$$x > \sqrt{2} \text{ or } x < -\sqrt{2}$$

Solve $g''(x) < 0$ to find intervals of convexity:

$$12x^2 - 24 < 0$$
$$12x^2 < 24$$
$$x^2 < 2$$
$$-\sqrt{2} < x < \sqrt{2}$$

So, the function $g(x) = x^4 - 12x^2$ is concave up (convex) on the intervals $x > \sqrt{2}$ and $x < -\sqrt{2}$ and concave down (concave) on the interval $-\sqrt{2} < x < \sqrt{2}$.

□□□